

CHARACTERIZING SLICES FOR PROPER ACTIONS OF LOCALLY COMPACT GROUPS

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ABSTRACT. In his seminal work [14], R. Palais extended a substantial part of the theory of compact transformation groups to the case of proper actions of locally compact groups. Here we extend to proper actions some other important results well known for compact group actions. In particular, we prove that if H is a compact subgroup of a locally compact group G and S is a small (in the sense of Palais) H -slice in a proper G -space, then the action map $G \times S \rightarrow G(S)$ is open. This is applied to prove that the slicing map $f_S : G(S) \rightarrow G/H$ is continuous and open, which provides an external characterization of a slice. Also an equivariant extension theorem is proved for proper actions. As an application, we give a short proof of the compactness of the Banach-Mazur compacta.

1. INTRODUCTION

The letter G will denote a Hausdorff topological group with unit element $e \in G$. All spaces are assumed to be completely regular and Hausdorff.

By an action of G on a space X we mean a continuous map $(g, x) \mapsto gx$ of the product $G \times X$ into X such that $ex = x$ and $(gh)x = g(hx)$, whenever $x \in X$, $g, h \in G$ and e is the unity of G . A space X together with a fixed action of the group G is called a G -space.

If X and Y are G -spaces, then a continuous map $f : X \rightarrow Y$ is called a G -map or an equivariant map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. For a point $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ of G is called the *stabilizer* or *isotropy subgroup* at x . Clearly, $G_x \subset G_{f(x)}$ whenever f is a G -map and $x \in X$.

If X is a G -space, then for a subset $S \subset X$ and a subgroup $H \subset G$, the H -hull (or H -saturation) of S is defined as follows: $H(S) = \{hs \mid h \in H, s \in S\}$. If S is the one point set $\{x\}$, then the H -hull $H(\{x\})$ usually is denoted by $H(x)$ and called the H -orbit of x . The orbit space X/H is always considered in its quotient topology. A

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subset $S \subset X$ is called H -invariant or, if it coincides with its H -hull, i.e., $S = H(S)$. A G -invariant set is also called, simply, invariant.

For a closed subgroup $H \subset G$, by G/H we will denote the G -space of cosets $\{xH \mid x \in G\}$ under the action induced by left translations, i.e., $g(xH) = (gx)H$ whenever $g \in G$ and $xH \in G/H$.

One of the most fundamental facts in the theory of G -spaces when G is a compact Lie group is, the so called, Slice Theorem. In its full generality it was proved by G. Mostow [11] (see also [13, Corollary 1.7.19]).

Theorem 1.1 (Slice Theorem). *Let G be a compact Lie group and X a G -space. Then for every point $x \in X$, there exists a G_x -slice $S \subset X$ such that $x \in S$.*

We recall the definition of an H -slice (cf. [13, §1.7]).

Definition 1.2. *Let X be a G -space and H a closed subgroup of G . A subset $S \subset X$ is called an H -slice in X , if:*

- (1) S is H -invariant, i.e., $H(S) = S$,
- (2) S is closed in $G(S)$,
- (3) if $G \setminus H$, then $gS \cap S = \emptyset$,
- (4) the saturation $G(S)$ is open in X .

If in addition $G(S) = X$, then we say that S is a global H -slice of X .

To each H -slice $S \subset X$, a G -map $f_S : G(S) \rightarrow G/H$, called the *slicing map*, is associated according to the following rule:

$$f_S(gs) = gH \quad \text{for every } g \in G, s \in S.$$

Lets check that that f_S is well defined. Indeed, if $gs = g's'$ for some $g, g' \in G$ and $s, s' \in S$ then $s = g^{-1}g's' \in S \cap g^{-1}g'S$. Then item (3) of Definition 1.2 yields that $g^{-1}g' \in H$ which is equivalent to $gH = g'H$, as required. Thus, the slicing map f_S is well defined.

It is immediate that the slicing map is equivariant, i.e., $f_S(gx) = gf_S(x)$ for all $x \in G(S)$ and $g \in G$. It is also clear that $S = f^{-1}(eH)$, where $eH \in G/H$ stands for the coset of the unit element $e \in G$.

It is a well known fact that the slicing map is continuous whenever the acting group G is compact (see e.g., [13, Theorem 1.7.7]); this gives the following important external characterization of an H -slice.

Theorem 1.3. *Let G be a compact group, H a closed subgroup of G , and X a G -space. Then there exists a one-to-one correspondence between all equivariant maps $f : X \rightarrow G/H$ and global H -slices S in X given by $f \mapsto S_f := f^{-1}(eH)$. The inverse correspondence is given by $S \mapsto f_S$, above defined.*

Below, in Theorem 2.2 we generalize Theorem 1.3 to the case of proper actions of arbitrary locally compact groups, which are an important generalization of actions of compact groups. This result is preceded by Theorem 2.1 which establishes some important properties of small global slices. Then these results are applied in Section 3 to orbit spaces of proper G -spaces. In Section 4 we prove an equivariant extension theorem which is further applied to get an equivariant extension of a continuous map defined on a small cross section. In the final Section 5 the results of Sections 2 and 3 are applied to give a short proof of the compactness of the Banach-Mazur compacta $BM(n)$, $n \geq 1$.

Recall that the concept of a *proper action* of a locally compact group was introduced in 1961 in the seminal work of R. Palais [14]. This notion allowed R. Palais to extend a substantial part of the theory of compact Lie transformation groups to non-compact ones. Perhaps, some detailed definitions are in order here.

Let G be a locally compact group and X a G -space. Two subsets U and V in X are called *thin* relative to each other [14, Definition 1.1.1], if the set

$$\langle U, V \rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$$

called *the transporter* from U to V , has compact closure in G . A subset U of a G -space X is called *G -small*, or just *small*, if every point in X has a neighborhood thin relative to U .

Definition 1.4 ([14]). *Let G be a locally compact group. A G -space X is called *proper*, if every point in X has a small neighborhood.*

Each orbit in a proper G -space is closed, and each stabilizer is compact [14, Proposition 1.1.4]. It is easy to check that: (1) the product of two G -spaces is proper whenever one of them is so; (2) the inverse image of a proper G -space under a G -map is again a proper G -space.

Important examples of proper G -spaces are the coset spaces G/H with H a compact subgroup of a locally compact group G . Other interesting examples the reader can find in [1], [5], [9] and [14].

Among other important results, Palais proved in [14] the following generalization of the Slice Theorem.

Theorem 1.5 (Slice Theorem for proper actions). *Let G be a Lie group (not necessarily compact) and X a proper G -space. Then for every point $x \in X$, there exist an invariant neighborhood U of x and an equivariant map $f : U \rightarrow G/G_x$ such that $x \in f^{-1}(eG_x)$.*

Since the distinguished point eG_x is a global G_x -slice for the proper G -space G/G_x , and since the inverse image of a slice is again a slice (see

[13, Corollary 1.7.8]), we infer that $S = f^{-1}(eG_x)$ is a G_x -slice (in the sense of Definition 1.2). Besides, S is a small subset of its saturation $G(S) = U$ since eG_x is a small subset of G/G_x . Therefore, at the first glance, it may seem that Theorem 1.5 establishes something more than the existence of a slice. However, this is not the case as it follows from our Theorem 2.2 below.

2. SOME IMPORTANT PROPERTIES OF SMALL SLICES

In this section we will prove the following two main results.

Theorem 2.1. *Let G be a locally compact group, X a proper G -space, H a compact subgroup of G , and S a global H -slice of X which is a small subset. Then*

- (1) *the restriction $f : G \times S \rightarrow X$ of the action is an open map.*
- (2) *the restriction $p : S \rightarrow X/G$ of the orbit map $X \rightarrow X/G$ is an open map.*

Proof. (1) Let O be an open subset of G and U be an open subset of S . It suffices to show that the set $OU = \{gu \mid g \in O, u \in U\}$ is open in X .

Define $W = \bigcup_{h \in H} (Oh^{-1}) \times (hU)$. Observe that

$$X \setminus OU = f((G \times S) \setminus W).$$

Indeed, since $OU = f(W)$ and $X = f(G \times S)$, the inclusion $X \setminus OU \subset f((G \times S) \setminus W)$ follows.

Let us establish the converse inclusion $f((G \times S) \setminus W) \subset X \setminus OU$.

Assume the contrary, that there exists a point $gs \in f((G \times S) \setminus W)$ with $(g, s) \in (G \times S) \setminus W$ such that $gs \in OU$. Then $gs = tu$ for some $(t, u) \in O \times U$. Denote $h = g^{-1}t$. One has

$$s = g^{-1}tu = hu$$

and

$$(g, s) = (tt^{-1}g, g^{-1}tu) = (th^{-1}, hu) \in (Oh^{-1}) \times (hU).$$

Since both s and u belong to S , and $s = hu$, by item (3) of Definition 1.2, we conclude that $h \in H$. Consequently, $(Oh^{-1}) \times (hU) \subset W$, yielding that $(g, s) \in W$, a contradiction. Thus, the equality $X \setminus OU = f((G \times S) \setminus W)$ is proved.

Now we observe that $(G \times S) \setminus W$ is closed in $G \times S$, and hence, in $G \times X$. Since S is a closed small subset of X , by [1, Proposition

1.4(c)], f is a closed map. This yields that the set $f((G \times S) \setminus W)$ is closed, and hence, OU is open in X , as required.

(2) Indeed, let U be an open subset of S . By item (1) of this theorem, the saturation $G(U)$ is open in X yielding that the intersection $G(U) \cap S$ is open in S . Since $p^{-1}(p(U)) = G(U) \cap S$ we conclude that $p(U)$ is open in X/G , as required. \square

This theorem is now applied to give an external characterization of a slice in a proper G -space.

Theorem 2.2. *Let G be a locally compact group, H a compact subgroup of G . Let X be a proper G -space and S a global H -slice of X which is a small subset in X . Then the slicing map $f_S : X \rightarrow G/H$ is continuous and open. If, in addition, S is compact then f_S is also closed.*

Conversely, if one has an equivariant map $f : X \rightarrow G/H$, then the inverse image $S = f^{-1}(eH)$ is a global H -slice which is a small subset of X , and $f_S = f$.

Proof. Let $\alpha : G \times S \rightarrow X$ be the restriction of the action $G \times X \rightarrow X$ and let $\pi : G \times S \rightarrow G$ denote the projection.

The quotient map $p : G \rightarrow G/H$, $p(g) = gH$, is open (and closed since H is compact) and it makes the following diagram commutative:

$$\begin{array}{ccc} G \times S & \xrightarrow{\pi} & G \\ \downarrow \alpha & & \downarrow p \\ X & \xrightarrow{f} & G/H. \end{array}$$

Since S is a closed small subset of X , Theorem 2.1(1) yields that α is an open map. Since π and p are continuous, the equality $f\alpha = p\pi$ implies that f is continuous.

Since the maps π and p are open and α is continuous, we infer that f is also open. If, in addition, S is compact then the map π is also closed, which yields that f is closed.

The converse assertion is immediate since the point $eH \in G/H$ is a small global H -slice for G/H and an inverse image of a small global H -slice is so (see [14, p. 10]). The equality $f_S = f$ is a simple verification. \square

Since each compact subset of a proper G -space is a small subset [14, p. 300], Theorem 2.2 has the following immediate corollary.

Corollary 2.3. *Let G be a locally compact group, H a compact subgroup of G . Let X be a proper G -space and S a compact global H -slice*

of X . Then the slicing map $f_S : X \rightarrow G/H$ is continuous, open and closed.

Remark 2.4. Theorem 2.2 is not valid if the H -slice $S \subset X$ is not a small subset. Here is a simple counterexample. Let $G = \mathbb{R}_+$, the multiplicative group of the positive reals and $X = \mathbb{R}^2 \setminus \{0\}$, the Euclidean plane without the origin. Consider the action $G \times X \rightarrow X$ defined by means of the ordinary scalar multiplication, i.e., if $\lambda \in G$ and $A = (x, y) \in X$, then $\lambda * A := \lambda A = (\lambda x, \lambda y)$. It is easy to see that this action is proper. Further, let

$$S := \left\{ \left(x, \pm \frac{1}{x} \right) \mid x \in \mathbb{R} \setminus \{0\} \right\} \cup \{(0, \pm 1), (\pm 1, 0)\}.$$

Then, clearly, S is a global H -slice of X with $H = \{1\}$, the trivial subgroup of G while S is not a small subset of X . Also it is easy to see that the corresponding slicing map $f_S : X \rightarrow G$ is not continuous.

It is interesting to notice that the unit circle $S = \{A \in X \mid \|A\| = 1\}$ is also a global H -slice for X . However, in this case the slicing map $f_S : X \rightarrow G$ is continuous because S , being compact, is a small subset and then Corollary 2.3 applies. Moreover, in this case $f_S(A) = \|A\|$, which clearly, is a continuous G -map.

3. ORBIT SPACES

Existence of slices facilitates the study of transformation groups since, for example, it enables the reduction of global questions about transformation groups to local ones. On the other hand, existence of global slices in proper G -spaces enables the reduction of studying the orbit space of a non-compact group action to that of a compact subgroup.

The following result for G a compact group can be found in [13, Proposition 1.7.6].

Theorem 3.1. *Let G be a locally compact group, H a compact subgroup of G . Let X be a proper G -space and S a small global H -slice of X . Then the inclusion $S \hookrightarrow X$ induces a homeomorphism of the orbit spaces S/H and X/G .*

Proof. Let $p : S \rightarrow X/G$ be the restriction of the orbit projection $X \rightarrow X/G$. Then, according to Theorem 2.1, p is continuous and open. Since p is constant on the H -orbits of S , it induces a continuous open map $p' : S/H \rightarrow X/G$.

It remains to show that p' is a bijection. Indeed, if $G(x) \in X/G$ is any point then $x = gs$ for some $g \in G$ and $s \in S$ since S is a global H -slice. Then, clearly, $p(s) = G(s) = G(x)$ showing that $p'(H(s)) = G(x)$. Thus, p' is surjective.

To see that p' is injective, assume that $p'(H(s)) = p'(H(s_1))$ for some $s, s_1 \in S$. Then $G(s) = G(s_1)$, yielding that $s = gs_1$ for some $g \in G$. But then, by item (3) of Definition 1.2, we get that $g \in H$ which implies that $H(s) = H(gs_1) = H(s_1)$, as desired. This completes the proof that $p' : S/H \rightarrow X/G$ is a bijection, and hence, a homeomorphism. \square

Since every compact subset of a proper G -space is small [14, p. 300], Theorem 3.1 has the following immediate corollary.

Corollary 3.2. *Let G be a locally compact group, H a compact subgroup of G . Let X be a proper G -space and S a compact global H -slice of X . Then the inclusion $S \hookrightarrow X$ induces a homeomorphism of the orbit spaces S/H and X/G .*

It turns out that in the presence of compactness of the global H -slice S , the other assumptions in Theorem 3.1 may essentially be weakened. Namely, the following version of Theorem 3.1 holds true.

Proposition 3.3. *Let G be any topological group and H a closed subgroup of G . Let X be a G -space and S a compact global H -slice of X . Then the inclusion $S \hookrightarrow X$ induces a homeomorphism of the orbit spaces S/H and X/G provided that X/G is Hausdorff.*

Proof. Let $p : S \rightarrow X/G$ and $p' : S/H \rightarrow X/G$ be as in the proof of Theorem 3.1. Since p' is a continuous bijection, it remains to show that it is a closed map. But this is due to the hypotheses since S/H is compact and X/G is Hausdorff. \square

4. EXTENSION TO AN EQUIVARIANT MAP

Let G be a locally compact group and $H \subset G$ a compact subgroup. If X is a proper G -space and S a global H -slice of X , then it is well known (cf. [14, Proposition 2.1.3]) that any H -equivariant map $f : S \rightarrow Y$ uniquely extends to a G -equivariant map $F : X \rightarrow Y$. This result can be generalized in the following manner (for compact group actions it was proved in [7, Ch. I, Theorem 3.3]).

Theorem 4.1. *Let G be a locally compact group acting properly on the space X , and let Y be any G -space. Let S be any closed small subset of X , and let $f : S \rightarrow Y$ be a continuous map such that whenever s*

and gs are both in S (for some $g \in G$), then $f(gs) = gf(s)$. Then f can be extended uniquely to an equivariant map $F : G(S) \rightarrow Y$.

Proof. For any $g \in G$ and $s \in S$, put $F(gs) = gf(s)$. To see that F is well defined let $gs = g's'$. Then $s = (g^{-1}g')s'$ so that $f(s) = f((g^{-1}g')s') = g^{-1}g'f(s')$, by assumption. Thus, $gf(s) = g'f(s')$, as desired.

To see that F is continuous, let (x_i) be a net in $G(S)$ converging to $x \in G(C)$. Then $x_i = g_i s_i$ and $x = gs$ for some $s, s_i \in S$ and $g, g_i \in G$. Thus,

$$(4.1) \quad g_i s_i \rightsquigarrow gs.$$

Since S is a small set, we can choose a neighborhood U of s such that the transporter $\langle S, U \rangle$ has compact closure in G . Then, by convergence, there exists an index i_0 such that $(g^{-1}g_i)s_i \in U$ whenever $i \geq i_0$. Hence $g^{-1}g_i \in \langle S, U \rangle$ for $i \geq i_0$. Since $\langle S, U \rangle$ has compact closure in G , by passing to a subnet we may assume that $g^{-1}g_i \rightsquigarrow h$ for some $h \in G$. Then $g_i \rightsquigarrow gh$ and

$$(4.2) \quad g_i^{-1} \rightsquigarrow h^{-1}g^{-1}.$$

Now, (4.1) and (4.2) imply that $s_i \rightsquigarrow h^{-1}s$. Since $s_i \in S$, by closedness of S we conclude that $h^{-1}s \in S$. Then by the hypothesis we have $f(h^{-1}s) = h^{-1}f(s)$ and by continuity of f we get that $f(s_i) \rightsquigarrow f(h^{-1}s) = h^{-1}f(s)$. Consequently,

$$F(x_i) = F(g_i s_i) = g_i f(s_i) \rightsquigarrow gh(h^{-1}f(s)) = gf(s) = F(gs) = F(x).$$

This proves the continuity of F , as desired. \square

Recall that a continuous map $s : X/G \rightarrow X$ is called a cross section for the orbit map $\pi : X \rightarrow X/G$ if the composition πs is the identity map of X/G . It is easy to see that the image $C := s(X/G)$ is closed in X (since X is Hausdorff). It turns out that if, in addition, C is any small subset of X , then it uniquely determines the cross section. Because of this fact, we shall use the term “cross section” for the closed image of a cross section.

More precisely, we have the following result.

Proposition 4.2. *Let X be a proper G -space with G a locally compact group and let $\pi : X \rightarrow X/G$ be the orbit map. Let C be a small closed subset of X touching each orbit in exactly one point. Then the map $s : X/G \rightarrow X$ defined by $s(\pi(x)) = G(x) \cap C$ is a cross section. Conversely, the image of any cross section is closed in X .*

Proof. We need to show that s is continuous. For this let $A \subset C$ be closed. Since C is a small closed subset, one can apply [1, Proposition 1.4(c)] according to which the set $s^{-1}(A) = G(A)$ is closed in X , as desired.

For the converse, let $C = s(X/G)$ and let (x_i) be a net in C converging to $x \in X$. We have $\lim p(x_i) = p(x)$ and $\lim s(p(x_i)) = s(p(x))$. Therefore, $x = \lim x_i = \lim s(p(x_i)) = s(p(x)) \in C$ showing that $x \in C$. Thus, C is closed. \square

Theorem 4.1 has the following interesting corollary.

Corollary 4.3. *Let G be a locally compact group, X a proper G -space and Y any G -space. Assume that $C \subset X$ is a closed small cross section of the orbit map $p : X \rightarrow X/G$. Then each continuous map $f : C \rightarrow Y$ such that $G_c \subset G_{f(c)}$ for all $c \in C$, has a unique extension to an equivariant map $F : X \rightarrow Y$.*

Proof. If c and gc belong to C for some $g \in G$, then $gc = c$ since C touches the orbit $G(c)$ in exactly one point. Thus, $g \in G_c$. Since $G_c \subset G_{f(c)}$, we get that $gf(c) = f(c) = f(gc)$. Thus, the hypotheses of Theorem 4.1 are fulfilled, and hence, its application completes the proof. \square

Remark 4.4. *The example of the proper G -space X in Remark 2.4 shows that in Proposition 4.2 one cannot omit the smallness condition on the set C .*

5. AN APPLICATION

In this section we apply Corollary 3.2 to give a short proof of the compactness of the Banach-Mazur compacta $BM(n)$, $n \geq 1$.

As usual, for an integer $n \geq 1$, \mathbb{R}^n denotes the n -dimensional Euclidean space with the standard norm, and $GL(n)$ denotes the real full linear group. We denote by $\mathcal{B}(n)$ the hyperspace of all compact convex bodies of \mathbb{R}^n with odd symmetry about the origin, equipped with the Hausdorff metric

$$d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\},$$

where d is the standard Euclidean metric on \mathbb{R}^n .

We consider the natural action of $GL(n)$ on $\mathcal{B}(n)$ defined as follows: $(g, A) \mapsto gA$; $gA = \{ga \mid a \in A\}$, for all $g \in GL(n)$, $A \in \mathcal{B}(n)$.

In [2] it was proved that the $GL(n)$ -space $\mathcal{N}(n)$ consisting of all norms $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, endowed with the compact-open topology and the natural action of $GL(n)$, is a proper $GL(n)$ -space. Using the fact that $\mathcal{B}(n)$ is $GL(n)$ -equivariantly homeomorphic to $\mathcal{N}(n)$ (see [3, p. 210]), we infer that $\mathcal{B}(n)$ is a proper $GL(n)$ -space [3]. A direct way of proving the properness of the $GL(n)$ -space $\mathcal{B}(n)$ one can find in [6, Theorem 3.3].

According to a theorem of F. John [8], for any $A \in \mathcal{B}(n)$, there is unique maximal volume ellipsoid $j(A)$ contained in A (respectively, minimal volume ellipsoid $l(A)$ containing A). Usually, $j(A)$ is called the John ellipsoid of A , and $l(A)$ is called the Löwner ellipsoid of A .

This fact allows to define two *special* global $O(n)$ -slices in $\mathcal{B}(n)$, where $O(n)$ denotes the orthogonal subgroup of $GL(n)$.

Denote by $J(n)$ the subset of $\mathcal{B}(n)$ consisting of all bodies $A \in \mathcal{B}(n)$ for which the ordinary Euclidean unit ball $\mathbb{B}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$ is the maximal volume ellipsoid *contained* in A . In [3, Theorem 4] it was proved that $J(n)$ is a *global* $O(n)$ -slice for $\mathcal{B}(n)$.

Analogously, the subspace $L(n)$ of $\mathcal{B}(n)$ consisting of all bodies $A \in \mathcal{B}(n)$ for which \mathbb{B}^n is the minimal volume ellipsoid *containing* A , is a *global* $O(n)$ -slice for $\mathcal{B}(n)$.

Lets give a direct proof that $J(n)$ and $L(n)$ are compact.

Proposition 5.1. *$J(n)$ is compact.*

Proof. It is known that there is a closed ball $D \subset \mathbb{R}^n$ centered at the origin such that $A \subset D$ for every $A \in J(n)$. Moreover, it was proved by F. John [8] that the radius of D may be taken even \sqrt{n} .

Thus $J(n)$ is a subset of the hyperspace $cc(D)$ of all non-empty compact convex subsets of D endowed with the Hausdorff metric topology. Since $cc(D)$ is compact (in fact, it is homeomorphic to the Hilbert cube [12, Theorem 2.2]), it suffices to show that $J(n)$ is closed in $cc(D)$. But this is evident since, if $(A_k)_{k \in \mathbb{N}} \subset J(n)$ is a sequence converging to $A \in cc(D)$, then A should contain the unit ball \mathbb{B}^n since every A_k does. Hence, A has non-empty interior, i.e., it is a convex body, and then, $A \in \mathcal{B}(n)$. Now we apply [3, Theorem 4(4)] according to which $J(n)$ is closed in $\mathcal{B}(n)$. This yields that $A \in J(n)$, and hence, $J(n)$ is closed in $cc(D)$, as desired.

□

In a similar way, one can prove the compactness of the global $O(n)$ -slice $L(n)$. Here one should take into account that every $A \in L(n)$ contains the closed ball of radius $\frac{1}{\sqrt{n}}$ centered at the origin of \mathbb{R}^n (see [8, p. 559]).

Thus, $\mathcal{B}(n)$ is a proper $GL(n)$ -space and $J(n)$ is a global $O(n)$ -slice of it (see [3, Theorem 4]). Since by Proposition 5.1, $J(n)$ is compact, Corollary 3.2 immediately yields the following corollary (cf. [3, Corollary 1]).

Corollary 5.2. *The orbit space $\mathcal{B}(n)/GL(n)$ is homeomorphic to the $O(n)$ -orbit space $J(n)/O(n)$.*

Corollary 2.3 and Proposition 5.1 yield the following corollary.

Corollary 5.3. *The slicing map $f_{J(n)} : \mathcal{B}(n) \rightarrow GL(n)/O(n)$ corresponding to the global $O(n)$ -slice $J(n)$ is continuous, open and closed.*

The compactness of the orbit space $\mathcal{B}(n)/GL(n)$ originally was established in [10]. Corollary 5.2 and Proposition 5.1 immediately yield an alternative and short proof of the compactness of $\mathcal{B}(n)/GL(n)$, known as the Banach-Mazur compactum $BM(n)$ (see [3]).

Corollary 5.4. *$\mathcal{B}(n)/GL(n)$ is a compact metrizable space.*

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